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Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential

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Abstract
We investigate some implications of the freezing scenario proposed by Carpentier and Le Doussal (CLD) for a random energy model (REM) with logarithmically correlated random potential. We introduce a particular (circular) variant of the model, and show that the integer moments of the partition function in the high-temperature phase are given by the well-known Dyson Coulomb gas integrals. The CLD freezing scenario allows one to use those moments for extracting the distribution of the free energy in both high- and low-temperature phases. In particular, it yields the full distribution of the minimal value in the potential sequence. This provides an explicit new class of extreme-value statistics for strongly correlated variables, manifestly different from the standard Gumbel class.

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1. Introduction

The random energy model (REM) introduced by Derrida in [1] is characterized by the partition function $Z_\beta = \sum_{i=1}^{M} e^{-\beta V_i}$, where β is the inverse temperature, and V_i for $i = 1, \ldots, M$ are random variables with the typical variance $\langle V_i^2 \rangle = \overline{V^2} = O(\log M)$ for $M \to \infty$. Such a model, as well as its generalized version (GREM) [2, 3] continues to play a paradigmatic role in statistical mechanics of disordered systems. Simple enough to allow for a detailed analytical investigation by various methods, the freezing transition exemplified by REM appears to be a rather generic phenomenon. It emerges with surprising regularity in a variety of physical situations, ranging from transparency of random media [4], directed polymers in random environment [5], p-spin glass models and the glass transition [6], random heteropolymers and models of protein folding [7] to properties of quantum particles in a random magnetic field [8, 9], and thermodynamics of a single particle in random Gaussian landscapes [10]. It is also a rich model from purely probabilistic point of view [11], and has an interesting dynamical counterpart: aging [12].

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As the low-temperature behaviour in statistical mechanics is obviously controlled by the lowest available energies, it is not surprising that a detailed description of the freezing phenomenon is intimately related to the so-called extreme-value statistics of random variables [13]. For independent, identically distributed variables V_i, the cumulative probability distribution $P_m(x)$ of $V_m = \min\{V_1, \ldots, V_M\}$ is well understood in mathematical literature and, provided the support of the distribution extends to $-\infty$ but decays faster than any power-law, is given in the limit $M \gg 1$ by the Gumbel law

$$P_m(x) = \text{Prob}(V_m > x) = \exp[-e^{(x+a_M)/b_M}], \tag{1}$$

where the constants a_M and b_M depend explicitly on the distribution of V_i, but the double-exponential shape of the Gumbel law is very robust (universal). This universality extends to a very broad class of *correlated* variables provided those correlations decay fast enough, see a detailed description in [14]. For a mathematically rigorous analysis of REM and GREM based, in particular, on the extreme-value statistics, see [15]; see also [16] for recent developments.

A few years ago Carpentier and Le Doussal (CLD) [9] studied a specific case of correlated random variables that is arguably the richest, most challenging, and relevant for applications. More precisely, thinking of the index i as referring to the sites of a certain lattice, the covariance $\langle V_i V_j \rangle$ considered by CLD depended on the distance $d(i, j)$ between those sites *logarithmically*. An independent support of the fact that logarithmically correlated potentials should play a special role was obtained recently through a thorough analysis of statistical mechanics of a single particle in high-dimensional random energy landscapes [10].

To understand the extreme-value statistics in the logarithmic case and to relate it to a REM-like freezing transition CLD developed a powerful, albeit non-rigorous, real space renormalization group approach to the distribution $\mathcal{P}(Z)$ of the partition function Z_β. In this way they discovered that logarithmic models represent, in a sense, continuous analogues of directed polymers on disordered trees, a result somewhat anticipated in [8]. The statistical mechanics of the directed polymer problem on a tree is known to be amenable to a travelling wave analysis, see the celebrated paper [5]. Similarly, for the logarithmic case the Laplace transform of the distribution $\mathcal{P}(Z)$ was also shown to satisfy a kind of travelling wave equation (see also works [17] for a relation between the travelling waves and the extreme-value statistics, in particular, in the context of the zero temperature directed polymer problem on a tree). Solutions of equations of that type are known to exhibit a characteristic change of the shape at some critical front velocity, and following [5] that change was interpreted by CLD as a signal of a REM-like freezing. In particular, the CLD analysis revealed that such a transition implies, among other properties, a universal *non-Gumbel* shape of the far-left tail for the cumulative distribution of the minimal value of logarithmically correlated variables: $P_m(x \to -\infty) \approx 1 - \text{const}|x| \, e^{ax}$ which is clearly different from the Gumbel tail $P_m(x \to -\infty) \approx 1 - \text{const} \, e^{ax}$.

Although CLD's renormalization group is able to predict successfully the universal far-tail features of the distribution expected to be shared by all logarithmically correlated potentials, the calculation of the full distribution for a *given* potential is beyond the scope of that method. Indeed, the renormalization procedure needs specifying what the authors called a 'fusion of environments' rule [9]. As CLD convincingly argue, the precise form of that rule is not important for recovering universal properties. But the actual shape of the travelling wave equation certainly depends on the particular fusion rule employed. As no guiding principle for the fusion rule selection had been provided, a more detailed analysis of specific models appears to be problematic.

The present paper grew out of attempts to overcome the above difficulty. Our main observation is that for a particular variant of the logarithmically-correlated REM a more

detailed analysis seems to be possible. In particular, we are able to conjecture the *full distribution function* $P_m(x)$ pertinent for the case considered. Namely, after appropriate rescaling the cumulative distribution $P_m(x)$ of the minimum turns out to be given by the central result of this work

$$P_m(x) = 2\,\mathrm{e}^{\frac{\beta_c x}{2}}\,K_1\big(2\,\mathrm{e}^{\frac{\beta_c x}{2}}\big), \tag{2}$$

where $K_1(x)$ stands for the modified Bessel (Macdonald) function, and the parameter β_c is given by the inverse transition temperature in the model. As such, the corresponding distribution is manifestly different from the standard Gumbel double-exponential form.

Our method draws its inspiration from CLD analysis and is essentially based on the pattern of the freezing transition revealed in [5, 9]. However, we do not resort to the renormalization group construction or travelling wave technique, and in this way circumvent the need to know the microscopic fusion rule. Instead, we demonstrate below that assuming CLD freezing scenario provides a way to extend the moments of the partition function from the high-temperature phase to the region below the transition. Roughly speaking, if those moments can be explicitly calculated above the transition—as is the case for both the standard REM as well as our variant of the logarithmic model—they can be used to conjecture the shape of the Laplace transform of the probability density above the transition point. Then the CLD approach can be used to recover the full distribution of the partition function/free energy below the transition, yielding in particular the extreme-value statistics.

The structure of the paper is the following. We start with a short general discussion of CLD freezing scenario, in particular its implications for the moments of the partition function in the low-temperature phase. As an illustration we show how CLD scenario can be used in the standard REM case to recover the exact low-temperature expressions for the partition function moments, obtained long ago by Gardner and Derrida via a rather tedious analysis [3]. After that we introduce and analyse the particular (circular) version of the one-dimensional logarithmically correlated REM.

2. CLD freezing scenario: general relations and implications

The central object of the subsequent analysis is the Laplace-transform $G_\beta(p)$ of the probability distribution $\mathcal{P}(Z_\beta)$ of the partition function: $G_\beta(p) = \langle \exp\{-pZ_\beta\}\rangle$. Here and henceforth the angular brackets stand for the expectation with respect to the distribution of random variables V_i. Our approach is based on the assumption that such a Laplace transform can be efficiently found in the high-temperature phase.

Following [5, 9] we introduce the variable x via $p = \mathrm{e}^{\beta x}$ and consider the function $\tilde{G}(x) = \langle \exp\{-\mathrm{e}^{\beta x} Z_\beta\}\rangle$. Extending the Derrida–Spohn scenario, CLD postulate that in the thermodynamic limit $M \gg 1$ the latter function has a shape of a travelling wave, that is

$$\tilde{G}(x) = g_\beta(x + m_\beta(L)), \qquad m_\beta(L \gg 1) \approx c(\beta)L + \text{l.o.t.}, \qquad L = \ln M \tag{3}$$

where we introduced the parameter L to identify with notations used in [9], and *l.o.t.* stands for the lower order terms when $L \to \infty$.[3] In general, both the travelling wave profile $g_\beta(y)$ and the wave velocity $c(\beta)$ depend on the inverse temperature β. The REM-like transition in this approach is described by a 'freezing' of both the velocity and the profile function at a certain transition temperature $\beta = \beta_c$ so that in the full low-temperature phase $\beta \geqslant \beta_c$ one has

$$g_\beta(y) = g_{\beta_c}(y), \qquad m_\beta(M \gg 1) \approx c(\beta_c)L + \text{l.o.t.} \tag{4}$$

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[3] Establishing the precise form of those terms, both above and below the transition, was one of the central points of CLD analysis. It is, however, of no direct relevance for us in the present paper.

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By the very definition of the function $\tilde{G}(x)$ we then have the following relation for $\beta \geqslant \beta_c$:

$$\langle \exp\{-e^{\beta x} Z_\beta\} \rangle = \tilde{G}(x)|_{\beta \geqslant \beta_c} = g_{\beta_c}(x + c(\beta_c)L) \qquad (5)$$

which fixes the shape of the Laplace transform $G_{\beta \geqslant \beta_c}(p) = \langle \exp\{-pZ_\beta\} \rangle$ in the whole temperature range below the transition point.

Equipped with such a scenario, we will exploit that the knowledge of the Laplace transform allows one to calculate the moments of the partition function below the transition by employing the standard identity:

$$\langle Z_\beta^{-\nu} \rangle = \frac{1}{\Gamma(\nu)} \int_0^\infty dp \, p^{\nu-1} \langle e^{-pZ_\beta} \rangle = \frac{1}{\Gamma(\nu)} \int_0^\infty dp \, p^{\nu-1} g_{\beta_c}\left(\frac{1}{\beta} \ln p + c(\beta_c)L\right), \qquad (6)$$

where $\Gamma(x)$ stands for the Euler Gamma function, and we used that $x = \frac{1}{\beta} \ln p$.

According to the freezing scenario, the mean value of the free energy in the low-temperature phase is to leading order in $L \gg 1$ temperature independent, and is given by $\langle F \rangle = -\frac{1}{\beta} \langle \ln Z_\beta \rangle = -c(\beta_c)L$. Being interested in the fluctuations of the free energy, we introduce the random variable $f = F - \langle F \rangle$ whose probability density we denote $\mathcal{P}_\beta(f)$. After changing the integration variable in (6) to $y = \frac{1}{\beta} \ln p + c(\beta_c)L$, introducing $s = -\beta\nu$ and integrating once by parts we observe that (6) takes the following form:

$$\langle e^{-sf} \rangle_f \equiv \int_{-\infty}^\infty e^{-sf} \mathcal{P}_\beta(f) \, df = -\frac{1}{\Gamma\left(1 - \frac{s}{\beta}\right)} \int_{-\infty}^\infty e^{-sy} \left[\frac{d}{dy} g_{\beta_c}(y)\right] dy. \quad (7)$$

Thus, the only function needed for investigating the low-temperature phase is the shape of the travelling wave profile $g_\beta(y)$ at the critical point $\beta = \beta_c$. For a general nonzero temperature $\beta_c < \beta < \infty$ one can extract the explicit form of the free-energy distribution $\mathcal{P}_\beta(f)$ by noting that the analytical continuation $s \to is$ converts the relation (7) to a Fourier transform which as we shall shortly see can be frequently inverted explicitly. A particular simple relation is obtained in the zero-temperature limit $\beta \to \infty$ where the free energy simply reduces to the minimum value of all random energies in the sample $F \to V_m = \min_i\{V_i\}$. It is immediately clear from (7) that

$$\lim_{\beta \to \infty} \mathcal{P}_\beta(f) = -\frac{d}{df} g_{\beta_c}(f) \qquad (8)$$

yielding a very general relation between the shape of the critical profile $g_{\beta_c}(x)$ and the probability density $\mathcal{P}_m(x) \equiv -\frac{d}{dx} P_m(x)$ of the fluctuations of the extreme values in the sample: $\mathcal{P}_m(x) = -\frac{d}{dx} g_{\beta_c}(x)$.

Let us briefly demonstrate how this method works for the standard REM with i.i.d. Gaussian sequence of V_i. Introducing the notation $Z^{(0)} = e^{(1+\frac{\beta^2}{\beta_c^2}) \ln M}$, the analysis of [9] demonstrated that everywhere in the high-temperature phase $\beta \leqslant \beta_c$ the Laplace transform is given by

$$G_{\beta < \beta_c}(p) = \int_0^\infty e^{-pZ} \mathcal{P}(Z) \, dZ \approx e^{-pZ^{(0)}}, \qquad 0 \leqslant pZ^{(0)} \ll O(\ln \ln M). \qquad (9)$$

Identifying $L = \ln M$ as in (3) we find from (9) and the correspondence $G(p)|_{p=e^{\beta x}} \equiv g_\beta(x + c(\beta)L)$ the REM travelling wave profile which is given by

$$g_\beta(y) = \exp\{-e^{\beta y}\}, \qquad c(\beta) = \frac{1}{\beta} + \frac{\beta}{\beta_c^2}. \qquad (10)$$

In particular, when approaching the transition point $\beta = \beta_c$ the shape and velocity of the travelling wave tends to the limiting values

$$g_{\beta_c}(y) = \exp\{-e^{\beta_c y}\}, \qquad c(\beta_c) = \frac{2}{\beta_c}. \qquad (11)$$

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According to the general discussion, the Laplace transform $\langle e^{-pZ_\beta} \rangle = G_\beta(p)$ of the probability distribution of the partition function in the low-temperature phase $\beta > \beta_c$ can be found from

$$\tilde{G}(x) = g_{\beta_c}(x + c(\beta_c)L) = \exp\{-e^{\beta_c x + 2L}\} \Rightarrow G_{\beta > \beta_c}(p) = e^{-C_M p^{\frac{\beta_c}{\beta}}}, \qquad C_M = e^{2L} = M^2, \tag{12}$$

where we again used the correspondence $x = \frac{1}{\beta} \ln p$. In turn, the last expression can be immediately employed to recover the (non-integer) moments of the partition function from the first of relations in (6). Substituting there $G_{\beta > \beta_c}(p)$ from (12) yields after a straightforward manipulation

$$\left\langle Z_\beta^{-\nu} \right\rangle = [Z_a]^{-\nu} \frac{\Gamma\left(1 + \frac{\beta}{\beta_c} \nu\right)}{\Gamma(1 + \nu)}, \qquad Z_a = e^{2\frac{\beta}{\beta_c} \ln M} \tag{13}$$

valid as long as $-\nu < \frac{\beta_c}{\beta}$. The latter moments are, to leading approximation, precisely those obtained by Gardner and Derrida [3], and coincide, as expected from general arguments, with the moments of a totally asymmetric Lévy stable distribution of index β_c/β [13, 15, 18]. Finally, the above moments allow one to recover the distribution of the free-energy fluctuations in the low-temperature phase of the REM, which seems not to be written explicitly in the literature. Namely, making an analytic continuation $\nu \to is/\beta$, and introducing $f = -\frac{1}{\beta} \ln Z/Z_a$ we note that (13) takes a form of the Fourier transform of the probability density for f, see (7). Inverting that transform gives

$$\mathcal{P}_{\beta > \beta_c}^{\text{REM}}(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isf} \frac{1}{\Gamma\left(1 + \frac{is}{\beta}\right)} \Gamma\left(1 + \frac{is}{\beta_c}\right) ds = -\frac{d}{df} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{e^{n\beta_c f}}{\Gamma\left(1 - n\frac{\beta_c}{\beta}\right)}. \tag{14}$$

In particular, the zero-temperature limit $\beta \to \infty$ coincides with the general relation (8) between $\mathcal{P}_m$ and g_{β_c}, as given by equation (11), which immediately yields the famous Gumbel distribution for the minimal energy. It is also evident that the far-left tail $f \to -\infty$ of the probability density $\mathcal{P}_\beta^{\text{REM}}(f)$ is of the Gumbel form everywhere in the low-temperature phase $\beta > \beta_c$, again expected from general arguments [13, 15, 18].

3. The circular logarithmic REM

3.1. Definition of the model and the moments in the high-temperature phase

Consider the lattice of M points positioned equidistantly at the circumference of a unit circle. Their angular coordinates are given by $\theta_k = \frac{2\pi}{M} k$, $k = 1, 2, \ldots, M$. With each point we associate a Gaussian random variable V_i, with position-independent variance $\langle V_i^2 \rangle = V^2$ and covariances chosen to be

$$C_{kl} = \langle V_k V_l \rangle = -g^2 \ln \left\{ 4 \sin^2 \frac{\theta_k - \theta_l}{2} \right\}. \tag{15}$$

For the consistency of the procedure we have to choose variance V^2 in a way ensuring positive definiteness of the full covariance matrix with entries $V^2 \delta_{kl} + (1 - \delta_{kl}) C_{kl}$. The condition amounts to $V^2 > -\lambda_{\max}$, with $\lambda_{\max}$ being the largest eigenvalue of the matrix $\hat{C}$ with entries C_{kl}. The matrix $\hat{C}$ is by definition a *circulant* real symmetric, with zero diagonal. Hence, its eigenvalues are given by $\lambda_q = \sum_{l=2}^{M} C_{1l} \omega_q^{l-1}$, where $\omega_q = \exp\left\{\frac{2\pi iq}{M}\right\}$ are roots of Mth degree from unity. The largest eigenvalue corresponds to $q = 0$, when $\omega_q = 1$. Then we have

$$\lambda_{\max} = -g^2 \sum_{s=1}^{M-1} \ln \left\{ 4 \sin^2 \frac{\pi s}{M} \right\} = -g^2 \ln \left\{ \prod_{s=1}^{M-1} 4 \sin^2 \frac{\pi s}{M} \right\}.$$

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Using the identity: $\frac{M}{2^{M-1}} = \prod_{s=1}^{M-1} \sin\frac{\pi s}{M}$ we see that $\lambda_{\max} = -2g^2 \ln M$, so we have to choose the variance to satisfy $V^2 = 2g^2 \ln M + W$, with an arbitrary positive $W > 0$. In what follows we assume that $W = O(1)$ when $M \gg 1$ and therefore can be safely neglected if we are interested only in the leading terms in the thermodynamic limit.

We define the partition function for our model in the standard way through $Z_\beta = \sum_{i=1}^{M} e^{-\beta V_i}$, with the goal to evaluate positive integer moments $\langle [Z_\beta]^n \rangle$ in the limit $M \gg 1$. The expected value of the partition function is obviously independent of the covariance and is given by the standard REM expression, $\langle Z_\beta \rangle = \exp\{\ln M[1 + \beta^2 g^2]\}$, so the first nontrivial moment is

$$\langle [Z_\beta]^2 \rangle = \sum_{i_1=1}^{M} \langle \exp\{-2\beta V_{i_1}\} \rangle + \sum_{i_1 \neq i_2}^{M} \langle \exp\{-\beta(V_{i_1} + V_{i_2})\} \rangle$$

$$= e^{\ln M(1+4\beta^2 g^2)} + e^{\ln M 2\beta^2 g^2} \sum_{i_1 \neq i_2}^{M} \left[2\sin\left(\frac{\pi(i_1 - i_2)}{M}\right) \right]^{-2\beta^2 g^2}. \qquad (16)$$

Introducing $\tilde{\theta}_k = \pi k/M$ for $k = 1, 2, \ldots, M$ we see that the second term in (16) is a kind of Riemann sum, and that in the limit $M \to \infty$ can be approximated as

$$\sum_{i_1 \neq i_2}^{M} \left[2\sin\left(\frac{\pi(i_1 - i_2)}{M}\right) \right]^{-2\beta^2 g^2} \approx \frac{M^2}{\pi} \int_{\pi/M}^{\pi - \pi/M} [2\sin\tilde{\theta}]^{-2\beta^2 g^2} \, d\tilde{\theta}$$

$$\approx \begin{cases} e^{2\ln M} \dfrac{\Gamma(1 - 2\beta^2 g^2)}{\Gamma^2(1 - \beta^2 g^2)}, & 2\beta^2 g^2 < 1 \\[3mm] e^{\ln M(1+2\beta^2 g^2)} \dfrac{1}{\pi^{2\beta^2 g^2}(2\beta^2 g^2 - 1)2^{2\beta^2 g^2 - 1}}, & 2\beta^2 g^2 > 1. \end{cases} \qquad (17)$$

The first line in (17) corresponds to the convergent integral in the limit $M \to \infty$ which can be easily evaluated by reducing it to the standard Euler's integral of the first kind , see [22], p 898. In contrast, the second line is obtained by extracting the leading term of the divergent integral. Comparing now second 'off-diagonal' term in (16) with the first 'diagonal' one, we find that for $2\beta^2 g^2 > 1$ the diagonal and off-diagonal contributions are of the same order, whereas for $2\beta^2 g^2 < 1$ the off-diagonal contribution dominates. Finally, we arrive at

$$\langle [Z_\beta]^2 \rangle |_{M \gg 1} \approx \begin{cases} e^{2\ln M(1+\beta^2 g^2)} \dfrac{\Gamma(1 - 2\beta^2 g^2)}{\Gamma^2(1 - \beta^2 g^2)}, & \beta^2 g^2 < 1/2 \\[3mm] e^{\ln M(1+4\beta^2 g^2)} \left[1 + \dfrac{1}{\pi^{2\beta^2 g^2}(2\beta^2 g^2 - 1)2^{2\beta^2 g^2 - 1}} \right], & \beta^2 g^2 > 1/2. \end{cases} \qquad (18)$$

The case of a general positive integer moment can be treated along the same lines. Denoting $x_i = e^{-\beta V_i}$ we use

$$\left(\sum_{i=1}^{M} x_i \right)^n = \sum_{p_1=0,\ldots,p_M=0}^{n} \frac{n!}{p_1! \ldots p_M!} x_1^{p_1} \ldots x_M^{p_M} \delta_{n, \sum_{i=1}^{M} p_i}$$

$$= \sum_{i_1=1}^{M} x_{i_1}^n + \sum_{l=1}^{n-1} \frac{n!}{l!(n-l)!} \sum_{i_1=1}^{M-1} \sum_{i_2=i_1+1}^{M} x_{i_1}^l x_{i_2}^{n-l} + \cdots, \qquad (19)$$

where in the second line we regrouped the terms according to partitions of the integer n into the sum of nonnegative integers with length k (i.e. the number of nonzero parts) taking values $k = 1, \ldots, n$. For example, partitions of the length $k = 1$ are sets $\{p_1, \ldots, p_M\}$ with all but one p_j equal to zero, and with the remaining nonzero integer taking the value $p = n$.

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The total contribution of those partitions is obviously $\sum_{i_1=1}^{M} x_{i_1}^{n}$ which is the first term in (19). Similarly, the contribution of all partitions of the length $k = 2$ is precisely the second term in the above expression, and so on. Finally, we perform the ensemble averaging of the above sum using the identity

$$\langle x_{i_1}^{l_1} x_{i_2}^{l_2} \cdots x_{i_k}^{l_k} \rangle = e^{\ln M \beta^2 g^2 \sum_{q=1}^{k} l_q^2} \prod_{p<q}^{k} \left[2 \sin \left(\frac{\pi (i_p - i_q)}{M} \right) \right]^{-2\beta^2 g^2 l_p l_q} \tag{20}$$

valid in the case of all different indices in the set $i_1, \ldots, i_k$. In the limit $M \to \infty$ we then find by inspection the dominating terms. After manipulations generalizing those we performed earlier for $n = 2$ case we find the following general expression:

$$\langle [Z_\beta]^n \rangle |_{M \gg 1} \approx \begin{cases} e^{n \ln M (1+\beta^2 g^2)} \mathcal{I}_n(\beta^2 g^2), & n < 1/\beta^2 g^2 \\ e^{\ln M (1+n^2 \beta^2 g^2)} O(1), & n > 1/\beta^2 g^2 \end{cases} \tag{21}$$

where

$$\mathcal{I}_n(\beta^2 g^2) = \frac{n!}{\pi^n} \int_0^\pi d\theta_1 \int_{\theta_1}^\pi d\theta_2 \int_{\theta_2}^\pi d\theta_3 \ldots \int_{\theta_{n-1}}^\pi d\theta_n \prod_{p<q}^{n} [2\sin(\theta_p - \theta_q)]^{-2\beta^2 g^2}. \tag{22}$$

The explicit expression for the factor $O(1)$ in the second line of (21) is rather complicated, but is actually not needed for our purposes. Finally, using the symmetry of the integrand in (22) and noting that $|e^{2i\theta_p} - e^{2i\theta_q}|^2 = 4\sin^2(\theta_p - \theta_q)$ one observes that (22) is a particular case of the so-called Morris integral (related to the famous Selberg integral) whose value was first conjectured by Dyson in his studies of the Coulomb gas problem (see a very informative historic account and further references in [19])

$$\mathcal{I}_n(\beta^2 g^2) = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_n \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^{-2\beta^2 g^2} = \frac{\Gamma(1 - n\beta^2 g^2)}{[\Gamma(1 - \beta^2 g^2)]^n}. \tag{23}$$

The integral is clearly finite provided $\beta^2 g^2 < 1/n \leqslant 1$, and is divergent otherwise. The condition $\beta g < 1$ defines the high-temperature phase of the model. Note a certain similarity between our calculations and those arising in the framework of the multifractal random walk model of Bacry, Muzy and Delour [20].

The crucial point of our analysis is the ability to offer the explicit form of the probability density $\mathcal{P}(Z)$ of the partition function $Z_\beta = Z > 0$ which precisely reproduces the expressions for the moments (21). It is given by

$$\mathcal{P}(Z) = \begin{cases} \mathcal{P}_<(Z), & Z < Z_* \\ \mathcal{P}_>(Z), & Z > Z_*, \end{cases} \tag{24}$$

where we defined $Z_* = e^{2\ln M}$ and introduced for $\beta g < 1$ the two functions:

$$\mathcal{P}_<(Z) = \frac{1}{Z} \frac{1}{\beta^2 g^2} \left(\frac{Z_e}{Z} \right)^{\frac{1}{\beta^2 g^2}} \exp - \left\{ \left(\frac{Z_e}{Z} \right)^{\frac{1}{\beta^2 g^2}} \right\}, \qquad Z_e = \frac{e^{\ln M (1+\beta^2 g^2)}}{\Gamma \left(1 - \beta^2 g^2 \right)}, \tag{25}$$

and

$$\mathcal{P}_>(Z) = \frac{M}{\sqrt{4\pi \ln M}} \frac{1}{\beta g} \frac{1}{Z} e^{-\frac{1}{4\ln M \beta^2 g^2} \ln^2 Z} f \left(\frac{1}{2} \frac{\ln Z}{\ln M} \right). \tag{26}$$

To understand the structure of $\mathcal{P}(Z)$ note that the growth rate of the moments in the second line of equation (21) dictates that the far tail of the distribution must be of a log-normal nature. This is exemplified by the choice (26) for $\mathcal{P}_>(Z)$. On the other hand, the first line in (21) and the expressions (23) yield the probability density of the form $\mathcal{P}_<(Z)$ in equation (25). The

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crossover value $Z = Z_*$ is determined from the requirement for the leading exponential terms in the two pieces of the probability density to match smoothly, i.e. $\mathcal{P}_<(Z_*) \approx \mathcal{P}_>(Z_*)$ for $M \gg 1$. The factor $f(x)$ in (26) is assumed to be of the order of unity when its argument is of the order of unity, and is otherwise left unspecified. Finally, we verify in the appendix that the choice of $P(z)$ in equations (24)–(26) ensures the required change in moments $\langle Z^n \rangle_{M \gg 1}$ to occur precisely at $n = 1/g^2\beta^2$.

At the next step we use the probability density (24) for evaluating the Laplace transform function $G_\beta(p)$ in the high-temperature phase. A somewhat lengthy but straightforward calculation reveals that the log-normal tail $\mathcal{P}_>(Z)$ gives for $M \gg 1$ a negligible relative contribution to the Laplace transform, as long as we keep finite the value $pZ_e < \infty$. Effectively, it means that for our goals we can assume the partition function Z_β to be distributed with the probability density $\mathcal{P}_<(Z_\beta)$ given in equation (25). After a simple transformation of variables this implies a rather simple asymptotic formula

$$G_\beta(p) = \langle e^{-pZ_\beta} \rangle|_{M \gg 1} \approx \int_0^\infty \exp(-t - pZ_e t^a)\, dt, \qquad a = \beta^2 g^2 < 1. \tag{27}$$

Using such an expression , we can, for example, easily calculate the mean logarithm of the partition function, hence the mean free energy

$$\langle \ln Z \rangle = \lim_{\epsilon \to 0} \left[\Gamma(\epsilon) - \int_0^\infty dp\, p^{\epsilon-1} \langle e^{-pZ_\beta} \rangle \right] = \ln Z_e - a\Gamma'(1),$$

so that the mean free energy is given by

$$\langle F \rangle = -\frac{1}{\beta} \langle \ln Z \rangle = -\left(\frac{1}{\beta} + \beta g^2 \right) \ln M - \frac{1}{\beta} \ln\left[\Gamma(1 - \beta^2 g^2) \right] - \beta g^2 \Gamma'(1), \qquad \beta^2 g^2 < 1 \tag{28}$$

The leading term yields the expected universal REM expression for the mean free energy valid in the high-temperature phase, the rest corresponds to system-specific corrections. Those corrections diverge logarithmically when approaching the critical temperature $\beta = 1/g = \beta_c$, signalling of the phase transition. Note that the same result for the free energy can be recovered by the standard replica trick using moments (21).

The fluctuations of the free energy around its mean value can be easily recovered as well, using the explicit form of the distribution $\mathcal{P}_<(Z_\beta)$. Namely, introducing

$$f = F - \langle F \rangle = -\frac{1}{\beta} \ln\{Z/Z_e\} + f_0, \qquad f_0 = -\frac{a}{\beta} \Gamma'(1) \tag{29}$$

equation (25) implies the following probability density in the high-temperature phase $\beta < \beta_c$:

$$\mathcal{P}_\beta(f) = \frac{\beta}{a} \exp\left\{ \frac{\beta}{a}(f - f_0) - e^{\frac{\beta}{a}(f - f_0)} \right\}. \tag{30}$$

According to our previous discussion, a central role is played by $\tilde{G}(x) = G_\beta(p = e^{\beta x})$. Identifying $L = \ln M$ we observe that $\tilde{G}(x) \equiv g_\beta(x + m_L)$ where $m_L = \frac{1}{\beta} \ln Z_e$. Using equation (28) we further see that

$$m_L|_{L \gg 1} \approx \frac{1 + a}{\beta} L + O(1) = c(\beta)L + O(1), \qquad c(\beta) = \frac{1}{\beta} + \frac{\beta}{\beta_c^2},$$

again in full agreement with CLD results in the high-temperature phase, with $c(\beta)$ interpreted as the travelling wave velocity, and the wavefront profile given by

$$g_\beta(y) = \int_0^\infty dt \exp\left\{ -t - \frac{e^{\beta y}}{t^a} \right\}, \qquad a = \beta^2 g^2 < 1. \tag{31}$$

3.2. Transition to the low-temperature phase in the circular logarithmic REM

To investigate the low-temperature phase for $\beta \geqslant \beta_c$ we rely upon the CLD freezing scenario. When approaching the transition point $a = \beta^2/\beta_c^2 = 1$ the profile (31) tends to a well-defined limit

$$g_{\beta_c}(y) = 2\,\mathrm{e}^{\frac{\beta_c y}{2}}K_1\big(2\,\mathrm{e}^{\frac{\beta_c y}{2}}\big), \tag{32}$$

where $K_1(x)$ is the modified Bessel (Macdonald) function. According to the freezing arguments this shape via relation (8) is translated into the extreme-value probability density (2), which is our central result. This expression is non-Gumbel as the cumulative distribution behaves for $x \to -\infty$ as $P_m^{\mathrm{CLM}}(x) \approx 1 + \beta_c x\,\mathrm{e}^{\frac{\beta_c x}{2}}$ in full agreement with the analysis of [9]. The opposite tail for $x \to \infty$ has a generalized Gumbel-like shape $P_m^{\mathrm{CLM}}(x) \propto \exp\big\{\frac{\beta_c x}{4} - 2\,\mathrm{e}^{\frac{\beta_c x}{2}}\big\}$. Note a certain similarity of these two asymptotes to those of the probability density for the magnetization in the low-temperature phase of the XY model [21].

Moreover, for all temperatures below the transition $\beta \geqslant \beta_c$ the value of the leading term in m_L and the shape of the wavefront profile should be frozen to the critical point values, i.e. those for $\beta = \beta_c$. Thus, to the leading order $m_L(\beta > \beta_c) = c(\beta_c)L \equiv m_L^*$,[4] whereas the profile $g_\beta(y)$ is given by equation (32) for any $\beta > \beta_c$. In the same way as in the REM case this fact allows one to extract the moments of the partition function everywhere in the low-temperature phase when $\tilde{G}(x) = g_{\beta_c}(x + m_L^*)$. Employing now the critical profile shape equation (32) and substituting $x = \frac{1}{\beta}\ln p$ we recover the Laplace transform $G_\beta(p)$ of the probability density of the partition function below the transition

$$\int_0^\infty \mathrm{d}Z_\beta \mathcal{P}_{\beta > \beta_c}(Z_\beta)\,\mathrm{e}^{-pZ_\beta} = G_{\beta > \beta_c}(p) = 2bp^{\frac{\gamma}{2}}K_1(2bp^{\frac{\gamma}{2}}), \quad b = \mathrm{e}^{\frac{\beta_c}{2}m_L^*}; \quad \gamma = \frac{\beta_c}{\beta} \leqslant 1. \tag{33}$$

This gives us the possibility of calculating negative moments of the partition function as

$$\langle Z^{-\nu}\rangle = \frac{1}{\Gamma(\nu)}\int_0^\infty \mathrm{d}p\,p^{\nu-1}G_{\beta > \beta_c}(p) = \frac{b^{-\frac{2}{\gamma}\nu}}{\gamma\Gamma(\nu)}\Gamma\left(1 + \frac{\nu}{\gamma}\right)\Gamma\left(\frac{\nu}{\gamma}\right), \qquad \nu > 0 \tag{34}$$

where we have used the identity [22]

$$\int_0^\infty p^\mu K_\nu(ap)\,\mathrm{d}p = 2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{35}$$

Substituting here the explicit values of γ and b, and changing $\nu \to -\nu$ we finally get

$$\langle Z^\nu\rangle = \mathrm{e}^{\beta\nu m_L^*}\frac{1}{\Gamma(1-\nu)}\Gamma^2\left(1 - \frac{\beta}{\beta_c}\nu\right). \tag{36}$$

Although we used $\nu < 0$ in the course of derivation, a slight modification of the above procedure, see [3], shows that the above expression is valid in a wider region, as long as $\nu < \beta_c/\beta < 1$.

The mean value of the free energy F in the low-temperature phase is found in a similar way and the leading order term is simply $\langle F\rangle = -\beta^{-1}\langle\ln Z\rangle = -m_L^*$. Introducing the probability density $\mathcal{P}_\beta(f)$ of $f = F + m_L^*$ we can now rewrite equation (36) as

$$\int_{-\infty}^\infty \mathrm{e}^{sf}\mathcal{P}_\beta(f)\,\mathrm{d}f = \frac{1}{\Gamma\left(1 + \frac{s}{\beta}\right)}\Gamma^2\left(1 + \frac{s}{\beta_c}\right), \qquad \mathrm{Re}\,s > -\beta. \tag{37}$$

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[4] As $O(1)$ terms in m_L above the transition diverge logarithmically when $\beta \to \beta_c$ it is natural to expect that at the transition point they should be replaced with *const* $\ln L$. Actually, the analysis of [9] predicts the precise value *const* $= 1/2$ at the transition point. Unfortunately, the verification of this interesting prediction goes beyond the precision of our analysis.

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In particular, similarly to the REM case after an analytic continuation $s \to \mathrm{i}s$ the probability density of the free energy for the circular logarithmic model (CLM) can be extracted for any $\beta > \beta_c$ by inverting the corresponding Fourier transform. The corresponding formula takes a form of an infinite series

$$\mathcal{P}_{\beta>\beta_c}^{\mathrm{CLM}}(f) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sf} \frac{1}{\Gamma\left(1+\frac{\mathrm{i}s}{\beta}\right)} \Gamma^2\left(1+\frac{\mathrm{i}s}{\beta_c}\right) \mathrm{d}s \tag{38}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}f}\left[1 + \sum_{n=1}^{\infty} \frac{\mathrm{e}^{n\beta_c f}}{n!(n-1)!\Gamma\left(1-n\frac{\beta_c}{\beta}\right)}\left(\beta_c f + \frac{1}{n} - 2\psi(n+1) + \frac{\beta_c}{\beta}\psi\left(1 - n\frac{\beta_c}{\beta}\right)\right)\right] \tag{39}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Exploiting the series expansion for the Macdonald function, see, e.g. p 909 of [22], it is easy to check that in the zero temperature limit $\beta \to \infty$ the free-energy distribution indeed reduces to the extreme-value probability density of form equation (2) in full agreement with the general relation (8). Equation (39) shows that the same non-Gumbel behaviour holds for the far-left tail $f \to -\infty$ of the free-energy distribution at any temperature below the transition.

3.3. Conclusion, discussions and open problems

In the present paper, we attempted to investigate some implications of the CLD freezing scenario [9] for a particular type of REM-like model with logarithmically correlated random potential on a circle. The chosen model seems to be especially attractive due to relatively simple expressions for the integer moments of the partition function in the high-temperature phase, given by the well-known Dyson Coulomb gas integral. We argue that in such a case the Laplace transform of the probability density of the partition function can be efficiently recovered. When combined with the freezing scenario this knowledge allows us to continue the Laplace transform to the low-temperature phase. We first check that the method indeed works for the standard REM example by recovering the well-known, yet nontrivial Gardner–Derrida formulae [3] for the moments of the partition function below the freezing point. The same method is then applied to the logarithmic model in question. In particular, we are able to recover the full distribution of the lowest minimum in the potential, equation (2), and this extreme-value statistics is manifestly non-Gumbel.

Although we think our results are supported by rather convincing arguments, the calculations are very essentially based on a few plausible but not yet fully verified assumptions. As such, mathematically our conclusions have the status of well-grounded conjectures. It would be certainly very desirable to find alternative ways of investigating the model, as well as to perform accurate numerical verification of the precise form of the extreme-value statistics. Another open problem is the universality of our result, equation (2), for logarithmically correlated random variables, in particular the shape of the right tail (see [17]). We hope our results provide enough incentive for further research in this direction.

Finally, it might be useful to provide an alternative view on our choice of the logarithmically correlated potential, equation (15). By employing the known identity: $-\ln\left(4\sin^2\frac{x_1-x_2}{2}\right) = 2\sum_{l=1}^{\infty}\frac{1}{l}\cos l(x_1-x_2)$ we see that the covariance function (15) represents, in fact, a 2π-periodic real-valued Gaussian random process $V(x) = \sum_{l=1}^{\infty}(v_l\,\mathrm{e}^{\mathrm{i}lx} + \bar{v}_l\,\mathrm{e}^{-\mathrm{i}lx})$ with a *self-similar* spectrum $\langle v_l \bar{v}_m \rangle = g^2 l^{-(2H+1)}\delta_{lm}$ characterized by the particular choice of the Hurst exponent $H = 0$. Such a process therefore represents a version of the so-called $1/f$ noise. To this end it is worth mentioning that the extreme-value statistics of the

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'roughness' associated with $1/f$ noise was investigated in [23], and found to be of Gumbel form.

Acknowledgments

Appendix

The positive integer moments of the distribution $\mathcal{P}(Z)$, see Equations (24)–(26), are given in the high-temperature phase $\beta g < 1$ by the sum of two terms

$$\langle Z^n \rangle = m_<^{(n)} + m_>^{(n)}. \tag{A.1}$$

The first contribution corresponding to equation (25) is given by

$$m_<^{(n)} = \int_0^{Z_*} \mathcal{P}_<(Z) Z^n \, dZ = Z_e^n \int_B^\infty \tau^{-\beta^2 g^2 n} e^{-\tau} \, d\tau, \qquad B = \left(\frac{Z_e}{Z_*} \right)^{\frac{1}{\beta^2 g^2}}. \tag{A.2}$$

In the limit $\ln M \to \infty$ we have from (24) and (25) $B \propto e^{-\frac{1}{\beta^2 g^2}(1-\beta^2 g^2)\ln M} \to 0$ in view of $\beta^2 g^2 < 1$. After a simple calculation we find

$$m_<^{(n)} = \begin{cases} Z_e^n \Gamma(1 - n\beta^2 g^2), & n < \dfrac{1}{\beta^2 g^2} \\[2ex] \dfrac{1}{\beta^2 g^2 n - 1} Z_e^{\frac{1}{\beta^2 g^2}} Z_*^{n - \frac{1}{\beta^2 g^2}}, & n > \dfrac{1}{\beta^2 g^2}. \end{cases} \tag{A.3}$$

As to the second contribution, a saddle-point analysis justified by $\ln M \gg 1$ shows that:

$$m_>^{(n)} = \int_{Z_*}^\infty e^{n \ln Z} \mathcal{P}_>(Z) \, dZ \approx \begin{cases} \dfrac{f(1)}{2\sqrt{\pi \ln M}(1 - n\beta^2 g^2)} e^{(1+2n-\frac{1}{\beta^2 g^2})\ln M}, & n < \dfrac{1}{\beta^2 g^2} \\[2ex] f(n\beta^2 g^2) e^{\ln M(1+\beta^2 g^2 n^2)}, & n > \dfrac{1}{\beta^2 g^2}, \end{cases} \tag{A.4}$$

Comparing the two contributions $m_>^{(n)}$ and $m_<^{(n)}$ within the high-temperature phase $\beta g < 1$ we see that

(1) $m_<^{(n)} \gg m_>^{(n)}$ as long as $1 < n < \frac{1}{\beta^2 g^2}$. Indeed

$$n(1 + \beta^2 g^2) - \left(1 + 2n - \frac{1}{\beta^2 g^2} \right) = (1 - \beta^2 g^2)\left(\frac{1}{\beta^2 g^2} - n \right) > 0$$

which implies

$$m_<^{(n)} \sim Z_e^n \sim e^{n(1+\beta^2 g^2)\ln M} \gg e^{(1+2n-\frac{1}{\beta^2 g^2})\ln M} \sim m_>^{(n)}.$$

(2) If $n > \frac{1}{\beta^2 g^2}$ we have $m_<^{(n)} \ll m_>^{(n)}$, as in this case

$$m_<^{(n)} \sim Z_e^{\frac{1}{\beta^2 g^2}} Z_<^{n - \frac{1}{\beta^2 g^2}} \sim e^{(1+2n-\frac{1}{\beta^2 g^2})\ln M} \ll m_>^{(n)} \sim e^{(1+\beta^2 g^2 n^2)\ln M},$$

which follows from

$$(1 + \beta^2 g^2 n^2) - \left(1 + 2n - \frac{1}{\beta^2 g^2} \right) = \left(\beta g n - \frac{1}{\beta g} \right)^2 > 0.$$

Accounting for equation (23) and the definition of Z_e in equation (25) we indeed see that the moments $\langle Z^n \rangle$ coincide for $\ln M \gg 1$ with the expressions for the partition function moments (21).

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